Moment vanishing of piecewise solutions of linear ODEs

Dmitry Batenkov and Gal Binyamini

Abstract We consider the “moment vanishing problem” for a general class of piecewise-analytic functions which satisfy on each continuity interval a linear ODE with polynomial coefficients. This problem, which essentially asks how many zero first moments can such a (nonzero) function have, turns out to be related to several difficult questions in analytic theory of ODEs (Poincare’s Center-Focus problem) as well as in Approximation Theory and Signal Processing (“Algebraic Sampling”). While the solution space of any particular ODE admits such a bound, it will in the most general situation depend on the coefficients of this ODE. We believe that a good understanding of this dependence may provide a clue for attacking the problems mentioned above.
In this paper we undertake an approach to the moment vanishing problem which utilizes the fact that the moment sequences under consideration satisfy a recurrence relation of fixed length, whose coefficients are polynomials in the index. For any given operator, we prove a general bound for its moment vanishing index. We also provide uniform bounds for several operator families.

1 Introduction

Let $f:[a,b]\to\mathbb{R}$ be a bounded piecewise-continuous function with points of discontinuity (of the first kind)

$$a=\xi_0<\xi_1<\dots<\xi_p<\xi_{p+1}=b,$$

Dmitry Batenkov
Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel. This author is supported by the Adams Fellowship Program of the Israel Academy of Sciences and Humanities.
e-mail: dima.batenkov@weizmann.ac.il

Gal Binyamini
Department of Mathematics, University of Toronto, Canada. e-mail: galbin@math.utoronto.ca

satisfying on each continuity interval $\left[\xi_j, \xi_{j+1}\right]$ a linear homogeneous ODE

$$\mathfrak{D} f \equiv 0, \tag{1}$$

where $\mathfrak{D}$ is a linear differential operator of order n with polynomial coefficients:

$$\mathfrak{D} = p_n(x)\partial^n + \dots + p_1(x)\partial + p_0(x)\mathrm{I}, \qquad \partial = \frac{\mathrm{d}}{\mathrm{d}x}, \deg\, p_j \leq d_j. \tag{2}$$

We say that such f belongs to the class $\mathscr{PD}(\mathfrak{D}, p)$. The union of all such $\mathscr{PD}(\mathfrak{D}, p)$ is the class $\mathscr{PD}$ of *piecewise D-finite functions*, which was first studied in [3].

Any $f \in \mathscr{PD}$ has finite moments of all orders:

$$m_k(f) = \int_a^b x^k f(x)\,\mathrm{d}x, \qquad k = 0, 1, 2, \dots \tag{3}$$

We consider the following questions.

Problem 1. Given $\mathfrak{D}$ and p, determine the *moment vanishing index* of $\mathscr{PD}(\mathfrak{D}, p)$, defined as

$$\sigma(\mathfrak{D}, p) \stackrel{\text{def}}{=} \sup_{f \in \mathscr{PD}(\mathfrak{D}, p),\ f \not\equiv 0} \{k : m_0(f) = \dots = m_k(f) = 0\} + 1.$$

In Theorem 3 below we shall prove that the moment vanishing index is always finite. Consequently, the following problem becomes meaningful.

Problem 2. Find natural families $\mathscr{F} \subset \mathscr{PD}$ which admit a uniform bound on the moment vanishing index, i.e. for which

$$\sigma(\mathscr{F}) = \sup_{\mathscr{PD}(\mathfrak{D}, p) \subset \mathscr{F}} \sigma(\mathfrak{D}, p) < +\infty.$$

Our main results, presented in Section 4, provide a general bound for $\sigma(\mathfrak{D}, p)$ in terms of $\mathfrak{D}$. As a result, several examples of families $\mathscr{F}$ admitting uniform bound as above are given. The main technical tool is the recurrence relation satisfied by the moment sequence, established previously in [3].

Our main application is the problem of reconstructing functions $f \in \mathscr{PD}$ from a finite number of their moments. Inverse moment problems appear in some areas of mathematical physics, for instance heat conduction and inverse potential theory [1, 9], as well as in statistics. One particular reconstruction technique, introduced in [3] and further extended to two-dimensional setting in [4], can be regarded as a prototype for numerous "algebraic" reconstruction methods in signal processing, such as finite rate of innovation [15] and piecewise Fourier inversion [2, 6]. These methods, being essentially nonlinear, promise to achieve better reconstruction accuracy in some cases (as demonstrated recently in [2, 6]), and therefore we believe their study to be important. In Section 2 below we show that an answer to Problem 2

would in turn provide a bound on the minimal number of moments (measurements) required for unique reconstruction of any $f \in \mathscr{F}$. In essence, the results of this paper can be regarded as a step towards understanding the range of applicability of the piecewise D-finite reconstruction method to general signals in $\mathscr{P}\mathscr{D}$. See Section 2 for further details.

Given a family $\mathscr{F} \subset \mathscr{P}\mathscr{D}$, consider the corresponding family of moment generating functions $\left\{I_f(z)\right\}_{f\in\mathscr{F}}$, where $I_f(z) = \sum_{k=0}^{\infty} m_k(f) z^{-k-1}$. Obtaining information on the moment vanishing index is in fact an essential step towards studying the analytic properties of I_f, in particular a bound on its number of zeros near infinity (as provided by the notion of "Taylor Domination", see [5]), as well as conditions for its identical vanishing. In turn, these questions play a central role in studies of the Center-Focus and Smale-Pugh problems for the Abel differential equation, see [7, 14] and references therein.

The moment vanishing problem has been previously studied in the complex setting by V.Kisunko [12]. He showed that a uniform bound $\sigma(\mathscr{F})$ exists for families $\mathscr{F}$ consisting of non-singular operators, by using properties of Cauchy type integrals. In contrast, in this paper we consider the real setting only, while proving uniform bounds for some singular (as well as regular) operator families. Our method is based on the linear recurrence relation satisfied by the moment sequence. Using this method, in Section 5 we provide an alternative proof of Kisunko's result, stating that the moment generating function $I_f(z)$ of some $f \in \mathscr{P}\mathscr{D}(\mathfrak{D}, p)$ satisfies a non-homogeneous ODE

$$\mathfrak{D} I_f(z) = R_f(z)$$

for a very special rational function $R_f(z)$, which depends on $\mathfrak{D}$ and on the values of f at the discontinuities.

In Section 6 we provide an interpretation of our main result in the language of Fuchsian theory of ODE.

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2 Moment reconstruction

We start by defining some preliminary notions.

Definition 1. The Pochhammer symbol $(i)_j$ denotes the falling factorial

$$(i)_j \stackrel{\text{def}}{=} i(i-1)\cdot\ldots\cdot(i-j+1), \qquad i \in \mathbb{R},\ j \in \mathbb{N}$$

and the expression $(i)_j$ is defined to be zero for $i<j$.

Definition 2. Given $\mathfrak{D}$ of the form (2), the *bilinear concomitant* ([11, p.211]) is the homogeneous bilinear form, defined for any pair of sufficiently smooth functions $u(x), v(x)$ as follows (all symbols depend on x):

$$\begin{aligned} P_{\mathfrak{D}}(u,v) \stackrel{\text{def}}{=} & u\left\{p_1 v - \partial(p_2 v) + \dots + (-1)^{n-1}\partial^{n-1}(p_n v)\right\} \\ &+ u'\left\{p_2 v - \partial(p_3 v) + \dots + (-1)^{n-2}\partial^{n-2}(p_n v)\right\} \\ &+ \dots \\ &+ u^{(n-1)}\cdot(p_n v). \end{aligned} \tag{4}$$

Proposition 1 (Green's formula, [11]). *Given $\mathfrak{D}$ of the form (2), let the formal adjoint operator be defined by*

$$\mathfrak{D}^*\{\cdot\} \stackrel{def}{=} \sum_{j=0}^{n} (-1)^j \partial^j \left\{p_j(x)\cdot\right\}.$$

Then for any pair of sufficiently smooth functions $u(x), v(x)$ the following identity holds:

$$\int_a^b v(x)(\mathfrak{D}u)(x)\,\mathrm{d}x - \int_a^b u(x)(\mathfrak{D}^* v)(x)\,\mathrm{d}x = P_{\mathfrak{D}}(u,v)(b) - P_{\mathfrak{D}}(u,v)(a). \tag{5}$$

Theorem 1 ([3]). *Let $f \in \mathscr{PD}(\mathfrak{D},p)$ with $\mathfrak{D}$ of the form (2). Denote the discontinuities of f by $a=\xi_0<\xi_1<\dots<\xi_p<\xi_{p+1}=b$. Then the moments $m_k=\int_a^b f(x)\,\mathrm{d}x$ satisfy*[1] *the recurrence relation*

$$\sum_{j=0}^{n}\sum_{i=0}^{d_j} a_{i,j}(-1)^j (i+k)_j\, m_{i-j+k} = \varepsilon_k, \quad k=0,1,\dots, \tag{6}$$

where

$$\varepsilon_k = -\sum_{j=0}^{p}\left\{P_{\mathfrak{D}}\left(f,x^k\right)\left(\xi_{j+1}^-\right) - P_{\mathfrak{D}}\left(f,x^k\right)\left(\xi_j^+\right)\right\}. \tag{7}$$

Proof. Apply Green's formula (5) to the identity

$$\int_{\xi_j}^{\xi_{j+1}} x^k(\mathfrak{D}f)(x)\,\mathrm{d}x \equiv 0$$

for each $j=0,\dots,p$ and sum up. The result is

[1] For consistency of notation, the sequence $\{m_k\}$ is understood to be extended with zeros for negative k.

$$\sum_{j=0}^{p}\int_{\xi_j}^{\xi_j+1} f(x)\mathfrak{D}^*\left\{x^k\right\}\mathrm{d}x = -\sum_{j=0}^{p}\left\{P_{\mathfrak{D}}\left(f,x^k\right)\left(\xi_{j+1}^-\right)-P_{\mathfrak{D}}\left(f,x^k\right)\left(\xi_j^+\right)\right\}$$
$$\int_a^b f(x)\mathfrak{D}^*\left\{x^k\right\}\mathrm{d}x = \varepsilon_k$$

The left-hand side of the last formula is precisely the linear combination of the moments given by the left-hand side of (6). This finishes the proof. □

Now consider the problem of recovering $f \in \mathscr{PD}(\mathfrak{D},p) \subset \mathscr{PD}$ from the moments $\{m_0(f),\dots,m_N(f)\}$ (the operator $\mathfrak{D}$ is assumed unknown in the most general setting). Based on the recurrence relation (6), we demonstrate in [3] that an exact recovery is possible, provided that the number $N \in \mathbb{N}$ is sufficiently large. However, the question of obtaining an upper bound for N turns out to be non-trivial, as we now demonstrate.

Definition 3. Given $\mathfrak{D}$ and p, the *moment uniqueness index* $\tau(\mathfrak{D},p)$ is defined by

$$\tau(\mathfrak{D},p) \stackrel{\text{def}}{=} \sup_{f,g\in\mathscr{PD}(\mathfrak{D},p),\ f\not\equiv g}\left\{k:\ m_j(f)=m_j(g),\ 0\leqslant j\leqslant k\right\}+1.$$

In other words, given $\mathfrak{D}$ and p, at least $\tau(\mathfrak{D},p)$ first moments of $f \in \mathscr{PD}(\mathfrak{D},p)$ are necessary for unique reconstruction of f.

Recalling boundedness of $\sigma(\mathfrak{D},p)$ (see Theorem 3 below), we immediately obtain the following conclusion.

Lemma 1. *For any operator $\mathfrak{D}$ and any p*

$$\tau(\mathfrak{D},p) \le \sigma(\mathfrak{D},2p).$$

Proof. Let $N=\sigma(\mathfrak{D},2p)$. Take f_1, f_2 having p jump points each, satisfying $\mathfrak{D} f_1 \equiv 0, \mathfrak{D} f_2 \equiv 0$ on each continuity interval such that

$$\begin{aligned} m_0(f_1) &= m_0(f_2) \\ &\dots \\ m_N(f_1) &= m_N(f_2). \end{aligned}$$

The function $g=f_1-f_2$ has at most $2p$ jumps, and it satisfies $\mathfrak{D}g \equiv 0$ on each continuity interval. The first N moments of g are zero, therefore $g \equiv 0$ and thus $f_1 \equiv f_2$. Therefore $\tau(\mathfrak{D},p) \le N$. □

Consequently, in order to uniquely reconstruct an unknown $f \in \mathscr{F} \subset \mathscr{PD}$, it is sufficient to get a uniform bound $\sigma(\mathscr{F})$ for the family $\mathscr{F}$. Perhaps the most natural choice for such families is when the parameters $p,n,\left\{d_j\right\}_{j=0}^{n}$ are fixed. Unfortunately, without making additional assumptions, the moment vanishing index of such families cannot be uniformly bounded. This can be seen from the following example.

Example 1. Let $\mathfrak{D}_m$ denote the Legendre differential operator

$$\mathfrak{D}_m = \left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} + m(m+1)\mathrm{I},$$

and consider $a=-1,\ b=1$ and $p=0$. It is well-known that for each $m \in \mathbb{N}$, the regular solution of $\mathfrak{D}_m f = 0$ is $\mathscr{L}_m$ - the Legendre polynomial of degree m. Since the first $m-1$ moments of $\mathscr{L}_m$ are zero, we conclude that

$$\sigma\left(\mathfrak{D}_m\right) = m$$

and therefore $\sigma(\mathfrak{D})$ cannot be uniformly bounded in terms of the combinatorial type of $\mathfrak{D}$ only.

Using the subsequent results, in Section 6 we shall in fact provide an explanation of this behaviour.

3 Generalized power sums

Proposition 2. *The sequence $\{\varepsilon_k\}$, given by Theorem 1, is of the form*

$$\varepsilon_k = \sum_{j=0}^{p+1}\sum_{\ell=0}^{n-1} \xi_j^{k-\ell}(k)_\ell\, c_{\ell,j}, \tag{8}$$

where each $c_{\ell,j}$ is a homogeneous bilinear form in the two sets of variables

$$\{p_m(\xi_j), p_m'(\xi_j), \dots, p_m^{(n-1)}(\xi_j)\}_{m=0}^n,$$
$$\{f(\xi_j^+) - f(\xi_j^-),\ f'(\xi_j^+) - f'(\xi_j^-), \dots, f^{(n-1)}(\xi_j^+) - f^{(n-1)}(\xi_j^-)\}.$$

Proof. Denote for convenience $f(a^-) = f(b^+) = 0$. Now consider the definition of $\{\varepsilon_k\}$ given by (7). Rearranging terms, we write

$$\varepsilon_k = \sum_{j=0}^{p+1}\left\{P_{\mathfrak{D}}\left(f,x^k\right)\left(\xi_j^+\right) - P_{\mathfrak{D}}\left(f,x^k\right)\left(\xi_j^-\right)\right\}.$$

Furthermore, using the fact that the functions $\{p_m(x)\}_{m=0}^n$ and x^k are continuous at each ξ_j, we have

$$\begin{aligned}P_{\mathfrak{D}}\left(f,x^{k}\right)\left(\xi_{j}^{+}\right)-P_{\mathfrak{D}}\left(f,x^{k}\right)\left(\xi_{j}^{-}\right) = & \\ & = \left\{f\left(\xi_{j}^{+}\right)-f\left(\xi_{j}^{-}\right)\right\}\times \\ & \quad \times\left\{p_{1}\left(\xi_{j}\right)\xi_{j}^{k}-\left(p_{2}\left(\xi_{j}\right)k\xi_{j}^{k-1}+p_{2}'\left(\xi_{j}\right)\xi_{j}^{k}\right)+\dots\right\} \\ & \quad +\dots \\ & \quad +\left\{f^{(n-1)}\left(\xi_{j}^{+}\right)-f^{(n-1)}\left(\xi_{j}^{-}\right)\right\}p_{n}\left(\xi_{j}\right)\xi_{j}^{k}.\end{aligned} \tag{9}$$

Now using the definition (4), the claim is evident. □

The expression (8) for ε_k is nothing else but a generalized power sum. Let us recall several well-known facts about them (see e.g. [8, Section 2.3] or [13]).

Proposition 3. *Let the sequence s_k be of the form*

$$s_k=\sum_{j=0}^{p+1}\sum_{\ell=0}^{n-1}a_{\ell,j}\,(k)_\ell\,\xi_j^{k-\ell}\qquad a_{\ell,j},\xi_j\in\mathbb{C}. \tag{10}$$

Then it satisfies the following linear recurrence relation with constant coefficients of length $n(p+2)+1$:

$$\left(\prod_{j=0}^{p+1}(\mathrm{E}-\xi_j\,\mathrm{I})^n\right)s_k=0 \tag{11}$$

where E *is the forward shift operator in k and* I *is the identity operator.*
Conversely, the fundamental set of solutions of the recurrence relation (11) *is*

$$\left\{\xi_0^k,k\xi_0^{k-1},\dots,(k)_{n-1}\,\xi_0^{k-n+1},\ \dots,\ \xi_{p+1}^k,k\xi_{p+1}^{k-1},\dots,(k)_{n-1}\,\xi_{p+1}^{k-n+1}\right\}.$$

Corollary 1. *The sequence s_k as above, which is not identically zero, can have at most $n(p+2)-1$ first consecutive zero terms $s_0=\dots=s_{n(p+2)-2}=0$.*

Proof. If $s_0=\dots=s_{n(p+2)-1}=0$, then by the recurrence relation (11) we would have automatically $s_{n(p+2)}=s_{n(p+2)+1}=\dots=0$. □

Corollary 2. *Assume that the numbers $\left\{\xi_j\right\}_{j=0}^{p+1}\subset\mathbb{C}$ are pairwise distinct. Let the sequence s_k be given by* (10), *with a-priori unknown $\left\{a_{i,j}\right\}$. If $s_k=0$ for all $k\in\mathbb{N}$, then necessarily all the coefficients $\left\{a_{i,j}\right\}$ are zero.*

4 Main results

Let us now return to our main goal, namely, obtaining upper bounds on the moment vanishing index $\sigma(\mathfrak{D},p)$.

Definition 4. Given $\mathfrak{D}$ of the form (2), denote for each $j=0,\dots,n$

$$\alpha_j \stackrel{\text{def}}{=} d_j - j,$$

and also

$$\alpha = \alpha(\mathfrak{D}) \stackrel{\text{def}}{=} \max_{j=0,\dots,n} \alpha_j.$$

Proposition 4. *Let $f \in \mathscr{P}\mathscr{D}(\mathfrak{D}, p)$. Then vanishing of the first $(p+2)n+\alpha(\mathfrak{D})$ moments of f (i.e. $m_0 = \cdots = m_{(p+2)n+\alpha-1} = 0$) implies identical vanishing of the sequence $\{\varepsilon_k\}$ defined by Theorem 1.*

Proof. Consider the recurrence relation (6). Denote its left-hand side by μ_k. Obviously, since each μ_k is a linear combination of the moments, we have

$$\mu_0 = \cdots = \mu_{n(p+2)-1} = 0.$$

Consequently, the corresponding right-hand sides also vanish, i.e.

$$\varepsilon_0 = \cdots = \varepsilon_{n(p+2)-1} = 0. \tag{12}$$

The conclusion follows immediately from Corollary 1. □

Now we establish our main result.

Theorem 2. *Let $f \in \mathscr{P}\mathscr{D}(\mathfrak{D}, p)$, $f \not\equiv 0$ with discontinuity points*

$$a = \xi_0 < \xi_1 < \cdots < \xi_p < \xi_{p+1} = b.$$

Assume that $p_n(\xi_j) \neq 0$ for at least one ξ_j as above. Then at most

$$(p+2)n + \alpha(\mathfrak{D}) - 1$$

first moments of f can vanish (i.e. $m_0 = \cdots = m_{(p+2)n+\alpha-2} = 0$).

Proof. Assume by contradiction that the first $(p+2)n+\alpha$ moments of f vanish, i.e.

$$m_0 = \cdots = m_{n(p+2)+\alpha-1} = 0.$$

By Proposition 4 and Corollary 2 we immediately conclude that

$$c_{\ell,j} = 0, \quad j = 0,\dots,p+1,\ \ell = 0,\dots,n-1,$$

where $\{c_{\ell,j}\}$ are described by Proposition 2. Now we take the concrete j for which $p_n(\xi_j) \neq 0$. This means that the operator $\mathfrak{D}$ is regular at ξ_j, and consequently each solution to $\mathfrak{D} f = 0$ in the neighborhood of ξ_j is uniquely determined by the initial values $f(\xi_j), \dots, f^{(n-1)}(\xi_j)$. We claim that

$$f\left(\xi_j^+\right)-f\left(\xi_j^-\right)=f'\left(\xi_j^+\right)-f'\left(\xi_j^-\right)=\cdots=f^{(n-1)}\left(\xi_j^+\right)-f^{(n-1)}\left(\xi_j^-\right)=0. \tag{13}$$

In this case, we would immediately conclude that the function f is analytic at ξ_j (being a solution of analytic ODE), contradicting the assumption that ξ_j is a point of discontinuity of f.

To prove (13), we proceed as follows. By Proposition 2 it is easy to see that the term $c_{n-1,j}(k)_{n-1}\xi_j^{k-n+1}$ is in fact equal to

$$\left\{f\left(\xi_j^+\right)-f\left(\xi_j^-\right)\right\}(k)_{n-1}\,p_n(\xi_j)$$

in the expression for ε_k. Since $p_n(\xi_j)\neq 0$, we conclude that $f\left(\xi_j^+\right)-f\left(\xi_j^-\right)=0$. Substituting this into (9), we see that the next term $c_{n-2,j}(k)_{n-2}\xi_j^{k-n+2}$ equals

$$\left\{f'\left(\xi_j^+\right)-f'\left(\xi_j^-\right)\right\}(k)_{n-2}\,\xi_j^{k-n+2}p_n(\xi_j),$$

and thus $f'\left(\xi_j^+\right)-f'\left(\xi_j^-\right)=0$. Proceeding in this manner, we arrive at (13). This finishes the proof of Theorem 2. □

As a first consequence, we have the real-valued version of the result by Kisunko [12].

Corollary 3. *For every $n,d>0$ and $p\geqslant 0$ consider the family*

$$\mathscr{F}^{(1)}_{n,p,d}=\left\{f\in\mathscr{PD}(\mathfrak{D},p):\quad \mathfrak{D}=\sum_{j=0}^{n}p_j(x)\,\partial^j,\ \alpha(\mathfrak{D})=d,\ p_n(x)\neq 0 \text{ on } [a,b]\right\}.$$

Then

$$\sigma\left(\mathscr{F}^{(1)}_{n,p,d}\right)\leqslant(p+2)n+d-1.$$

Since the leading coefficient $p_n(x)$ cannot vanish at more than $\deg p_n$ points, we also have the following result.

Corollary 4. *For every $n,d>0$ and $p\geqslant 0$ consider the family*

$$\mathscr{F}^{(2)}_{n,p,d}=\left\{f\in\mathscr{PD}(\mathfrak{D},p):\quad \mathfrak{D}=\sum_{j=0}^{n}p_j(x)\,\partial^j,\quad \alpha(\mathfrak{D})=d,\ \deg p_n<p+2\right\}.$$

Then

$$\sigma\left(\mathscr{F}^{(2)}_{n,p,d}\right)\leqslant(p+2)n+d-1.$$

Let us now try to establish what happens in the general case. Let $f\in\mathscr{PD}(\mathfrak{D},p)$, $f\not\equiv 0$. Consider two possibilities.

1. The sequence $\{\varepsilon_k\}$ does not vanish identically. In this case, at least some of its initial terms $\left\{\varepsilon_0,\dots,\varepsilon_{n(p+2)-1}\right\}$ must be nonzero (Corollary 1). But this immediately implies that some of the first $n(p+2)+\alpha-1$ moments must be nonzero as well (otherwise the equality (6) cannot hold).
2. The sequence $\{\varepsilon_k\}$ vanishes identically, but Theorem 2 is not applicable. In this case the recurrence relation (6) becomes homogeneous. We rewrite it in the form

$$\sum_{\ell=-n}^{\alpha} q_\ell(k)\, m_{k+\ell}=0, \qquad k=0,1,\dots, \tag{14}$$

where

$$q_\ell(k) \overset{\text{def}}{=} \sum_{j=0}^{n} (-1)^j a_{\ell+j,j}\,(k+\ell+j)_j\,. \tag{15}$$

The leading coefficient $q_\alpha(k)$ may have positive integer zeros. Let $\Lambda(\mathfrak{D})$ denote the largest such zero. Then we claim that no more than $\alpha+\Lambda(\mathfrak{D})$ moments can vanish. Indeed, starting with $k=\Lambda(\mathfrak{D})+1$ we can safely divide the recurrence (14) by $q_\alpha(k)$ and obtain

$$m_{k+\alpha}=\sum_{\ell=-n}^{\alpha-1} r_\ell(k)\, m_{k+\ell}, \quad k\geqslant \Lambda(\mathfrak{D})+1,$$

where $r_\ell(k)$ are some rational functions with non-vanishing denominators. Therefore if the first $\Lambda(\mathfrak{D})+\alpha+1$ moments are zero, then all the rest of the moments must vanish, implying vanishing of f itself.

Thus we have proved the following result.

Theorem 3. *For every $\mathfrak{D}, p$ we have*

$$\sigma(\mathfrak{D},p)\leqslant \max\{n(p+2)-1,\ \Lambda(\mathfrak{D})\}+\alpha(\mathfrak{D})\,.$$

In Section 6, we demonstrate that in the case of Fuchsian differential operators, the number $\Lambda(\mathfrak{D})$ has a well-known interpretation.

5 Moment generating function

In this section we provide an alternative proof for the result of Kisunko [12] concerning moment generating functions.

Proposition 5. Let $f\in\mathscr{P}\mathscr{D}$. *The formal power series*

$$I_f(z) \overset{def}{=} \sum_{k=0}^{\infty}\frac{m_k}{z^{k+1}}$$

is in fact the Laurent series of the Cauchy type integral

$$\int_a^b \frac{f(t)\,\mathrm{d}t}{z-t}.$$

Proof. Write $\frac{1}{z-t} = \frac{1}{z}\left(\frac{1}{1-\frac{t}{z}}\right)$ and expand into geometric series. Convergence follows immediately for $z \to \infty$. □

The generalized power sums (Section 3) also have a well-known interpretation as the Taylor coefficients of rational functions. The following fact is well-known, and so we omit the proof.

Proposition 6. *Let the sequence* $\{s_k\}$ *be of the form* (10)*. Then the formal generating function*

$$g(z) = \sum_{k=0}^{\infty} \frac{s_k}{z^{k+1}}$$

is a regular at infinity rational function, with poles $\left\{\xi_0,\dots,\xi_{p+1}\right\}$*, each with multiplicity at most n. In particular,*

$$g(z) = \sum_{j=0}^{p+1}\sum_{\ell=0}^{n-1} \frac{(-1)^{\ell}\,\ell!\,a_{\ell,j}}{(z-\xi_j)^{\ell+1}}. \tag{16}$$

Theorem 4. *Let* $f \in \mathscr{P}\mathscr{D}(\mathfrak{D},p)$*. Then the Cauchy integral* I_f *satisfies in the neighborhood of* ∞ *the inhomogeneous ODE*

$$\mathfrak{D}\, I_f(z) = R_f(z), \tag{17}$$

where $R_f(z)$ *is the rational function whose Taylor coefficients at infinity are given by the sequence* ε_k *as in* (7)*. Consequently,* $R_f(z)$ *is given by the explicit expression* (16)*, with* $a_{\ell,j}$ *replaced by* $c_{\ell,j}$ *from* (8) *(Proposition 2).*

Proof. Consider the asymptotic expansion of the function $\mathfrak{D}\, I_f$ at infinity

$$\mathfrak{D}\, I_f = \sum_{k=0}^{\infty} \frac{s_k}{z^{k+1}}.$$

By substituting $\mathfrak{D}$ as in (2) and $I_f = \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}}$ we get

$$\begin{aligned}
\mathfrak{D}\, I_f &= \sum_{j=0}^{n} p_j(z)\, I_f^{(j)}(z) = \sum_{j=0}^{n}\sum_{i=0}^{d_j}\sum_{k=0}^{\infty} m_k \frac{(-1)^j (k+j)_j}{z^{k+1+j-i}} a_{i,j} \\
{\scriptstyle (k+j-i\to t)} &= \sum_{t=0}^{\infty} \frac{1}{z^{t+1}} \sum_{j=0}^{n} (-1)^j a_{i,j} (t+i)_j\, m_{t+i-j} \\
&= \sum_{s=0}^{\infty} \frac{s_k}{z^{k+1}}.
\end{aligned}$$

Comparing powers of z we have that $s_k = \mu_k$ where μ_k denote the left-hand side of (6). From $\mu_k = \varepsilon_k$ the conclusion follows. $\square$

6 Fuchsian operators

In this section we employ notions from the classical Fuchsian theory of linear ODEs in the complex domain (we used the reference [10]).

Assume that the sequence $\{\varepsilon_k\}$ vanishes identically. In this case, the Cauchy integral I_f satisfies in the neighborhood of ∞ the *homogeneous ODE*

$$\mathfrak{D}\, I_f = 0.$$

Definition 5. The operator $\mathfrak{D}$ is said to belong to the class $\mathfrak{R}$ if it has at most a regular singularity at ∞.

Lemma 2. *Let $\mathfrak{D} \in \mathfrak{R}$. Then*

1. *The numbers α_j (see Definition 4) satisfy*

$$\alpha_n \geqslant \alpha_j, \quad j = 0, \dots, n-1. \tag{18}$$

2. *The characteristic exponents of $\mathfrak{D}$ at the point ∞ are the roots of the equation*

$$q_{\alpha_n}(s-1) = 0,$$

where $q_\ell(k)$ is defined by (15)*.*

Proof. Dividing the coefficients of $\mathfrak{D}$ by p_n, we get the operator

$$\partial^n + r_1(z)\,\partial^{n-1} + \dots + r_n(z)\,\mathrm{I}, \qquad r_j(z) = \frac{p_{n-j}(z)}{p_n(z)}.$$

A necessary and sufficient condition for the point at infinity to be at most a regular singularity of this operator is that the function $r_j(z)$ is analytic at ∞ and has a zero there of order at least j ([10, Theorem 9.8b]). That is,

$$\deg p_n - \deg p_{n-j} \geq j.$$

But this is equivalent to

$$\begin{gathered}\deg p_n - n \geqslant \deg p_{n-j} - (n-j)\\ \alpha_n \geqslant \alpha_{n-j}.\end{gathered}$$

To prove the second statement, substitute the formal Frobenius series at infinity

$$g(z)=\sum_{k=0}^{\infty}\frac{b_k}{z^{s+k}}$$

into $\mathfrak{D}g=0$. By complete analogy with the calculation in Theorem 4 we get the recurrence relation

$$\sum_{j=0}^{n}\sum_{i=0}^{d_j}(-1)^j a_{i,j}(t+s+i-1)_j b_{t+i-j}=0, \qquad t=0,1,\dots.$$

For $t=0$ we find the highest order coefficient in this recurrence to be equal to $(i-j=\alpha=\alpha_n)$

$$\sum_{j=0}^{n}(-1)^j(s+\alpha+j-1)_j a_{j+\alpha,j}=q_\alpha(s-1).$$

The proof is finished. □

Together with Theorem 3, this immediately implies the following bound.

Corollary 5. *Let $\mathfrak{D}\in\mathfrak{R}$, and let $\lambda(\mathfrak{D})$ denote its largest positive integer characteristic exponent at the point ∞. Then $\Lambda(\mathfrak{D})=\lambda(\mathfrak{D})-1$, and consequently*

$$\sigma(\mathfrak{D},p)\leqslant\max\{(p+2)n,\lambda(\mathfrak{D})\}+d_n-n-1.$$

Now let us return to Example 1. The following fact is well-known (e.g. [10, Section 9.10].

Proposition 7. *The Legendre differential operator $\mathfrak{D}_m$ is of Fuchsian type with singularities $-1,1,\infty$. The characteristic exponents at ∞ are $m+1$ and $-m$.*

Theorem 2 is clearly not applicable. Using the formula (7), it is easy to see that

$$P_{\mathfrak{D}}\left(f,x^k\right)(1)=P_{\mathfrak{D}}\left(f,x^k\right)(-1)=0$$

for any $f\in\mathscr{P}\mathscr{D}$, and therefore the sequence $\{\varepsilon_k\}$ in this case is identically zero. Consequently, we conclude that

$$\sigma(\mathfrak{D}_m,0)=m,$$

as expected.

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